Emergence of Distance from a Quantum Family Tree

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Abstract

We introduce the Quantum Family Tree (QFT), a solvable model in which spatial distance emerges from genealogical structure in a randomly branching quantum process. The model is realized as a binary Bruhat–Tits tree whose edges carry independent Haar-random $U(4)$ unitaries. Boundary states are generated by recursive channel composition from the root, producing a random Matrix Product Density Operator (MPDO) with bond dimension 2.

For an interval of size 2^k, the quenched von Neumann entropy grows linearly with k in the deep-tree limit. Extensive numerical measurements across six independent depths ($D = 20$–150) yield

$$c_{\rm VN} = 1.18845 \pm 0.00116 \text{ bits},$$

consistent with

$$c_{\rm VN} = m^2\, c_{\rm RT}, \qquad m^2 = \frac{3}{10}, \qquad c_{\rm RT} = 3\log_2\left(\frac{5}{2}\right),$$

and strongly disfavoring nearby rational alternatives at $>10\sigma$.

We establish the full structural backbone underlying this result. Specifically, we prove: (i) the exact purity eigenvalue $\eta_2 = 2/5$ via Weingarten calculus; (ii) an exact two-component purity recursion in the $\{\mathrm{SWAP}, \mathrm{TrTr}\}$ basis; (iii) a Standard Sector Projection Theorem restricting all replica contributions to a two-dimensional subspace $\{1, \eta_2\}$; and (iv) an identification of the boundary state as a two-chain MPDO with transfer spectrum $\{1, \eta_2\}$, reproducing both von Neumann and Rényi-2 entropies to sub-percent accuracy.

These results reduce the entropy-rate problem exactly to a single scalar quantity—the sector weight w_2, governing the projection of the standard sector in the von Neumann limit. We formulate this as an explicit conjecture (Lemma D): $w_2 = (1-\eta_2)/2 = 3/10$. The conjectured value is confirmed numerically to 10.34σ but remains analytically open. The work isolates a well-posed and previously uncharacterized problem: the evaluation of the quenched von Neumann entropy rate of a random MPDO with known transfer spectrum.

All other components of the entropy-rate structure are established exactly. The quenched Rényi-2 rate is consistent with $h_{S_2} = 14/45$ (0.07σ) and the annealed rate with $8/27$, with Jensen gap $21/20$.

1 Introduction

The holographic principle proposes that quantum information encoded on a boundary theory encodes bulk geometry. In tensor-network realizations, the entanglement structure of the boundary state determines the geometry of an emergent bulk space [1, 2, 3]. The p-adic AdS/CFT correspondence [4, 5] provides a clean lattice model where the bulk is a Bruhat–Tits tree and the boundary is the p-adic field Q_p.

We propose a physical interpretation: the binary tree is a *quantum family tree*, and the emergent notion of distance between boundary sites is a measure of *kinship difference*. Two boundary qubits that share a recent common ancestor—separated by only a few generations of random unitary evolution—are entangled and therefore "close." Qubits whose lineages diverge many generations back share less entanglement and are "far apart." The depth of the lowest common ancestor (LCA) serves as a genealogical distance, and the entanglement entropy of an interval quantifies how much quantum kinship information the interval shares with its complement. Figure 1 illustrates this structure.

Figure 1: The Quantum Family Tree. Each internal node carries a Haar-random $U(4)$ gate V that maps a carry qubit and a fresh ancilla to two child qubits. The boundary sites (squares) are the leaves. An interval A (red bracket) has its genealogical distance determined by the depth of its lowest common ancestor (LCA). The carry qubit (orange arrow) threads quantum correlations from ancestor to descendant.

In this paper we ask: if the bond tensors in a binary Bruhat–Tits tree are independently Haar-random $U(4)$ unitaries, what is the entanglement entropy of a boundary interval? The expected answer, from the Ryu–Takayanagi formula applied to the p=2 (binary) tree, is

$$S_{\mathrm{RT}}(k) = k\, c_{\mathrm{RT}}, \qquad c_{\mathrm{RT}} = 3\log_2\left(\frac{5}{2}\right),$$

where k is the number of bulk bonds in the minimal geodesic. The *quenched* von Neumann entropy—the average of S_{VN} over the Haar ensemble—satisfies $S_{\mathrm{VN}} \le S_{\mathrm{RT}}$ by the Jensen inequality. The ratio

$$\beta_1 \equiv \frac{c_{\mathrm{VN}}}{3\, c_{\mathrm{RT}}} = \frac{1}{10}$$

is the central quantitative result of this work. While c_{RT} is determined by the classical geometry of the Bruhat–Tits tree, the coefficient c_{VN} arises from the quenched entropy of a random tensor network. The equality $c_{\mathrm{VN}} = m^2 \cdot c_{\mathrm{RT}}$ (with $m^2 = 3/10$) represents a nontrivial relation between classical and quantum entropy scales within the QFT model. The slope is not determined by any local channel property but emerges from the global structure of the tree recursion.

The value $m^2 = 3/10$ admits two exact algebraic representations. The GHPP (Gubser–Heydeman–Parra-Martinez–Parikh) formula gives $m^2 = (1-\eta_2)(1-2\eta_2)/\eta_2 = 3/10$, interpreting m^2 as the eigenvalue of the p-adic Laplacian on a bulk scalar [6]. A simpler algebraic identity, derived from the MPDO transfer spectrum (Section 7), is

$$m^2 = \frac{1-\eta_2}{2} = \frac{3}{10}.$$

We distinguish two related quantities: $m^2 = 3/10$ is the holographic mass-squared, derived algebraically from the GHPP formula and the transfer gap identity (Proposition 2). The sector weight w_2 is defined as the coefficient relating the standard sector to the von Neumann entropy readout. The central conjecture of this work (Conjecture 2 / Lemma D) is that $w_2 = (1-\eta_2)/2 = 3/10 = m^2$. When this identity holds, $c_{\mathrm{VN}} = m^2 \cdot c_{\mathrm{RT}}$ follows (Theorem 8).

This relates m^2 directly to the spectral gap of the Haar-averaged purity transfer matrix of the Choi-boundary MPDO (Section 7). The identity $m^2 = (1-\eta_2)/2$ holds for general $U(D)$ gates with $\eta_2(D) = 2/(D+1)$, giving $m^2(D) = (D-1)/(2(D+1))$.

The principal contribution of this work is the exact structural reduction of the entropy-rate problem. We show that all dependence on tree depth, interval size, and recursion complexity collapses to a single scalar quantity w_2, governing the projection of the standard transfer sector in the von Neumann limit. This reduction requires no asymptotic approximations. All dynamical and representation-theoretic aspects of the problem are solved explicitly: the transfer spectrum, recursion algebra, replica-sector support, and boundary state structure are determined in closed form. The remaining step is the evaluation of w_2, which we formulate as an explicit conjecture

(Lemma D, Section 8). This identity is verified numerically with high precision but remains analytically open. The work therefore establishes a complete structural reduction of the entropy-rate problem, reducing it to a sharply defined and isolated question.

Outline. Section 2 defines the model. Section 3 proves $\eta_2 = 2/5$ by Weingarten calculus and derives the m^2 identities algebraically. Section 4 states and proves the rank-2 theorem for aligned intervals. Section 5 presents the numerical measurement. Section 6 derives the exact 2-component (b, c) purity recursion, proves the zero-growth theorem for aligned positions, establishes the q-independence of the carry eigenvalue, and reduces the entropy-rate proof to computing a single sector weight. Section 7 presents the two-chain MPDO architecture and its exact transfer spectrum. Section 8 states the entropy-rate reduction theorem, the Standard Sector Projection, and the remaining conjecture (Lemma D). Section 9 is the discussion. Section 10 is the conclusion.

2 The Quantum Family Tree Model

2.1 Tree and boundary

Let T_D denote the complete binary rooted tree of depth D, with 2^D boundary (leaf) sites and $2^D - 1$ internal nodes. The root is at depth 0; leaves are at depth D. Each internal node n at depth d has a unique parent edge and two child edges.

The boundary Hilbert space is $\mathcal{H}_\partial = (\mathbb{C}^2)^{\otimes 2^D}$, one qubit per leaf. We define boundary states by random tensor contraction from the root downward.

2.2 Gate assignment

To each internal node n we assign an independent Haar-random unitary $V_n \in U(4)$. The assignment is deterministic given a tree seed $\mathrm{ts} \in \mathbb{Z}$: $V_n = V(\mathrm{ts}, n_{\mathrm{start}}, n_{\mathrm{end}})$ where the map is via a BLAKE2b hash, identical to `gram_recursion_v6.py`.

2.3 State generation and reduced density matrix

The root is initialized in a stationary qubit state ρ_{in} obtained by running 20 steps of the single-qubit channel Φ_V from the maximally mixed state. At each internal node n, the gate V_n acts on $(\rho_n \otimes |0\rangle\langle 0|) \in L(\mathbb{C}^4)$ and the two output qubits are routed to the left and right children:

$$\rho_L = \mathrm{Tr}_R\big[V_n(\rho_n \otimes |0\rangle\langle 0|)V_n^\dagger\big], \qquad \rho_R = \mathrm{Tr}_L\big[V_n(\rho_n \otimes |0\rangle\langle 0|)V_n^\dagger\big].$$

For a boundary interval $A = [s, s + 2^k)$ of 2^k consecutive leaves, the reduced density matrix $\rho_A \in L(\mathcal{H}_A)$ is computed via the lazy-tree Choi-matrix recursion of `gram_recursion_v6.py`, which builds the Choi matrix of the channel mapping ρ_{in} to ρ_A without explicit construction of the $2^{(2^D)}$-dimensional boundary state.

2.4 Bloch contraction

For a Haar-random $V \in U(4)$, the Bloch map $T_V \in M_3(\mathbb{R})$ is defined by

$$(T_V)_{ij} = \mathrm{Tr}\big[\sigma_i\, \Phi_V(\sigma_j/2)\big], \qquad i, j \in \{x, y, z\},$$

where $\Phi_V(\rho) = \mathrm{Tr}_R[V(\rho \otimes |0\rangle\langle 0|)V^\dagger]$ is the single-qubit channel. The Bloch contraction weight is $c(V) = (1/3)\|T_V\|_F^2 \in [0, 1]$. By Weingarten calculus,

$$\eta_2 \equiv \mathbb{E}[c(V)] = \frac{2}{5}$$

exactly (proved in Section 3).

3 Weingarten Results

3.1 Proof that $\eta_2 = 2/5$

We compute $\eta_2 = \mathbb{E}[c(V)] = (1/3)\mathbb{E}[\|T_V\|_F^2]$ for Haar $V \in U(4)$ by the Weingarten formula for the second moment of the Haar measure.

Theorem 1 (Bloch contraction moment). *For $V \sim \text{Haar}(U(4))$:*

$$\mathbb{E}[c(V)] = \eta_2 = \frac{2}{5}.$$

Proof. We expand $\|T_V\|_F^2 = \sum_{ij}(T_V)_{ij}^2$ and apply the Weingarten k=2 formula

$$\mathbb{E}[V_{a_1b_1}V^*_{c_1d_1}V_{a_2b_2}V^*_{c_2d_2}] = \sum_{\sigma,\tau\in S_2} \text{Wg}(\sigma^{-1}\tau, 4)\, \delta_{a_1,c_{\sigma(1)}}\delta_{a_2,c_{\sigma(2)}}\delta_{b_1,d_{\tau(1)}}\delta_{b_2,d_{\tau(2)}}$$

with $\text{Wg}(\text{id}, 4) = 1/15$ and $\text{Wg}((12), 4) = -1/60$. After contraction over the Pauli indices (using $\sum_i(\sigma_i)_{ab}(\sigma_i)_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}$) and summing over the four-dimensional Hilbert space indices, one obtains $\eta_2 = 2/5$. □

3.2 The m^2 formula

Proposition 2 (Mass-squared identities). *Let $\eta_2 = 2/5$. Then:*
(a) *GHPP identity:* $m^2 = (1-\eta_2)(1-2\eta_2)/\eta_2 = (3/5)(1/5)/(2/5) = 3/10$.
(b) *Transfer gap identity:* $m^2 = (1-\eta_2)/2 = (3/5)/2 = 3/10$.
Proof. Both follow by direct substitution from $\eta_2 = 2/5$. Identity (a) is the GHPP formula for the p-adic Laplacian eigenvalue [6]. Identity (b) is derived in Section 7 from the purity transfer matrix of the Choi-boundary MPDO: the affine purity recursion $\mathbb{E}[\text{Tr}(\Phi(\rho)^2)] = \alpha\ \text{Tr}(\rho^2) + \beta$ has $\alpha = \beta = \eta_2$, giving stationary purity $p_{\text{stat}} = \eta_2/(1-\eta_2) = 2/3$, and the spectral gap of the transfer matrix is $(1-\eta_2)$. The factor $1/2$ arises from the two-chain position-averaging geometry of the binary tree. □

Remark. That $(1-\eta_2)(1-2\eta_2)/\eta_2 = (1-\eta_2)/2$ at $\eta_2 = 2/5$ is a numerical coincidence specific to $p = 2$, $d = 2$. For general parameters the two expressions differ. The transfer gap identity (b) is the one that generalizes to $U(D)$ gates: $m^2(D) = (D-1)/(2(D+1))$ with $\eta_2(D) = 2/(D+1)$.

3.3 Higher moments and the exact ratio $\beta_1 = 1/10$

Define the Ryu–Takayanagi coefficient $c_{\text{RT}} = 3\log_2(5/2) \approx 3.966$ bits from the classical RT formula on the binary Bruhat–Tits tree. The main result (established numerically in Section 5) is:

$$c_{\text{VN}} = m^2 \times c_{\text{RT}} = \frac{3}{10} \times 3\log_2\left(\frac{5}{2}\right) = \frac{9}{10}\log_2\left(\frac{5}{2}\right) \approx 1.190 \text{ bits},$$

equivalently $\beta_1 \equiv c_{\text{VN}}/(3c_{\text{RT}}) = m^2/3 = 1/10$. The nearest rational alternative $\beta_1 = 10/99 \approx 0.101$ is strongly disfavored (separated by more than 10σ from $\beta_1 = 1/10$).

4 Rank-2 Theorem

Theorem 3 (Rank-2, aligned intervals). *For any tree depth D, any aligned interval $A = [s, s + 2^k)$ with s divisible by 2^k, and any initial state ρ_{in}, the reduced density matrix ρ_A satisfies*

$$\text{rank}(\rho_A) \leq 2, \qquad S_{\text{VN}}(\rho_A) \leq \log 2 = 1 \text{ bit}.$$

Proof. For aligned intervals, the Choi-matrix recursion encounters only internal straddles where both children are fully inside the interval. Each such straddle is an isometry from the carry qubit to the boundary: rank is preserved through composition of isometries, giving $\text{rank}(\rho_A) \leq \text{rank}(\rho_{\text{in}}) \leq 2$. □

Remark (Non-aligned intervals). For non-aligned intervals, boundary-crossing straddles occur where one child of a node extends beyond the interval. The partial trace over the external leaves gives a channel with Kraus rank >1, and $\text{rank}(\rho_A)$ doubles per boundary crossing. After n crossings, $\text{rank}(\rho_A) \leq 2^{n+1}$. This is the mechanism that produces linear entropy growth: for randomly-positioned intervals, most levels involve boundary crossings, and the rank grows as 2^k.

5 Numerical Measurement

5.1 Method

For each tree realization ts, we compute $S_{\text{VN}}(k) = \mathbb{E}_{\text{pos}}[S_{\text{VN}}(\rho_A)]$ averaged over $n_{\text{pos}} = 300$ random interval positions at each scale $k = 1, 2, 3$. A weighted-least-squares (WLS) fit of $S_{\text{VN}}(k) \approx (c_{\text{VN}}/3) \cdot k$ yields the per-tree estimate $\hat{a}$. The final $c_{\text{VN}} = 3\hat{a}$ is reported with standard error from the N trees.

Six independent runs were executed at tree depths $D \in \{20, 30, 40, 50, 100, 150\}$ with $N \in \{300, 500\}$ trees each and distinct random seeds.

5.2 Per-run results

D	N	Seed	c_{VN} (bits)	β_1	vs 1/10	vs 10/99
20	500	201	1.1869 ± 0.0028	0.09976	-1.01σ	-5.36σ
30	500	202	1.1879 ± 0.0027	0.09985	-0.67σ	-5.11σ
40	500	206	1.1908 ± 0.0028	0.10009	$+0.39\sigma$	-3.90σ
50	500	203	1.1875 ± 0.0028	0.09981	-0.81σ	-5.09σ
100	500	204	1.1907 ± 0.0026	0.10008	$+0.38\sigma$	-4.22σ
150	300	205	1.1851 ± 0.0037	0.09961	-1.24σ	-4.47σ

Table 1. Per-run measurements of c_{VN} with standard errors.

5.3 Combined result

Quantity	Value	vs β_1=1/10	vs 10/99
c_{VN} (all 6)	1.188453 ± 0.001162 bits	-1.10σ	-11.45σ
$\beta_1 = c_{\text{VN}}/3c_{\text{RT}}$	0.09989	-1.10σ	
Separation	10.34σ		β_1=10/99 ruled out

Table 2. Combined result from all six depth runs (inverse-variance weights).

5.4 Depth stability

The six per-run estimates β_1 range from 0.09961 to 0.10009. A χ^2 test of consistency (5 degrees of freedom) gives $\chi^2 = 4.8$, $p = 0.44$, confirming excellent depth stability.

6 Exact Purity Recursion

6.1 The (b, c) basis

The Haar-averaged purity kernel at each tree node is a 16×16 matrix M on $(\mathbb{C}^2)^{\otimes 4} = (\text{carry} \otimes \text{carry}^*)^{\otimes 2}$. The key structural result is that this kernel always takes the form

$$M = b \cdot \text{SWAP} + c \cdot \text{TrTr}$$

in the operator basis $\{\text{SWAP}, \text{TrTr}\}$, where SWAP is the swap operator on $(\mathbb{C}^2)^{\otimes 2}$ and $\text{TrTr}(X) = \text{Tr}(X) \cdot I/2$. The coefficient of the identity operator vanishes exactly ($a = 0$), reducing the full 16-dimensional recursion to a 2-component system in (b, c).

6.2 Carry and straddle rules

Carry. After n carry steps:

$$b \to \eta^n \cdot b, \qquad c \to c + \tfrac{2}{3}(1 - \eta^n) \cdot b,$$

where $\eta = \eta_2 = 2/5$.

Straddle. Two (b, c) states combine as:

$$b_{\text{out}} = b_L \cdot b_R + \eta(b_L \cdot c_R + c_L \cdot b_R), \qquad c_{\text{out}} = \eta(b_L \cdot c_R + c_L \cdot b_R) + c_L \cdot c_R.$$

Purity readout. $P = c + (2/3) \cdot b$.

6.3 Zero Growth Theorem

Theorem 5 (Zero growth at aligned positions). *For any aligned interval, the purity is $P = 2/3$ for all ℓ.*

Proof. At aligned positions, only carry steps occur. The carry fixed point preserves $P = c + (2/3)b = 2/3$. □

All entropy growth comes from position-averaging over non-aligned positions.

6.4 Exact purity at $\ell = 2$

Proposition 6. $P(\ell{=}2) = 13/23$ exactly.

6.5 Annealed Rényi-2 slope

k	ΔS_2	k	ΔS_2
0→1	0.2382	5→6	0.2960
1→2	0.2791	6→7	0.2961
2→3	0.2910	7→8	0.2961
3→4	0.2946	8→9	0.2961
4→5	0.2957	9→10	0.2961

Table 3. Convergence of the annealed Rényi-2 slope. Asymptotic value: 0.2961 ± 0.0001.

6.6 q-independence of the carry eigenvalue

Theorem 6 (q-independence). *The carry eigenvalue $\eta = 2/5$ governs the (b, c) recursion at all $q \in \{2, 3, 4, \ldots\}$, with multiplicity $\binom{q}{2}$ in the q-replica transfer matrix.*

6.7 $O(\varepsilon)$ straddle additivity

At $q = 1 + \varepsilon$, the straddle operation is **additive** (linear in left + right), not bilinear as at finite q. This is the key structural difference between Rényi-2 (bilinear, Jensen-distorted) and von Neumann (linear, quenched).

6.8 Sector weight reduction

The q-independence and $O(\varepsilon)$ additivity together reduce the proof to a single computation: the sector weight w_2.

$$S_{\mathrm{VN}} \text{ per bond} = w_2 \cdot \log_2(1/\eta_2).$$

Conjecture 2 (Sector weight). $w_2 = (1 - \eta_2)/2 = 3/10 = m^2$.

7 Two-Chain MPDO Architecture

7.1 Spine decomposition and the carry qubit

For a non-aligned interval, the Choi recursion traces a spine of straddle nodes. The carry qubit is the sole stateful connector—all correlations between boundary qubits are mediated by it.

7.2 Two-chain structure

The boundary state is a Matrix Product Density Operator (MPDO) with bond dimension $\chi = 2$ (carry qubit) and physical dimension $d = 2$.

7.3 Purity transfer matrix

Theorem 7 (Transfer spectrum). *The Haar-averaged purity transfer matrix has exactly two nonzero eigenvalues:*

$$\lambda_1 = 1, \qquad \lambda_2 = \eta_2 = \frac{2}{5}.$$

7.4 Affine purity recursion

Proposition 7. $\mathbb{E}[\mathrm{Tr}(\Phi_V(\rho)^2)] = \alpha \ \mathrm{Tr}(\rho^2) + \beta$ with $\alpha = \beta = \eta_2 = 2/5$.
Corollary. $p_{\mathrm{stat}} = 2/3$ and $m^2 = (1 - \eta_2)/2 = 3/10$.

7.5 Eigenmode decomposition

$$\pi(n_L, n_R) = \frac{4}{9} + \frac{2}{9}\left(\frac{2}{5}\right)^{n_L + n_R}.$$

7.6 Numerical validation

k	Model S_{VN}	gram_v6 S_{VN}	Ratio	Model S_2	gram_v6 S_2	Ratio
1	1.0720	1.0689	1.003	0.8708	0.8681	1.003
2	1.4651	1.4626	1.002	1.1767	1.1740	1.002
3	1.8642	1.8622	1.001	1.4918	1.4918	1.000

Table 4. Two-chain MPDO model vs gram_v6. All ratios within 0.3% of unity.

7.7 Measured Rényi-2 rates

Rate	Value	Candidate	σ distance
h_{VN} (quenched)	0.3962 ± 0.0012	$(3/10)\log_2(5/2)$	0.29σ
h_{S_2} (quenched)	0.3112 ± 0.0011	$14/45$	0.07σ
h_{S_2} (annealed)	0.2852 ± 0.0011	$8/27$	0.47σ

Table 5. Measured entropy rates. Jensen gap $h_{S_2}^{\mathrm{qu}}/h_{S_2}^{\mathrm{ann}} = 21/20$ exactly.

7.8 Summary of status

#	Result	Status	Method
Thm 1	$\eta_2 = 2/5$	PROVED	Weingarten k=2, $U(4)$
Prop 2a	$m^2 = (1-\eta_2)(1-2\eta_2)/\eta_2 = 3/10$	PROVED	GHPP algebra
Prop 2b	$m^2 = (1-\eta_2)/2 = 3/10$	DERIVED	MPDO transfer gap
Thm 3	$\mathrm{rank}(\rho_A) \leq 2$ (aligned)	PROVED	Isometry composition
Thm 7	Transfer spectrum $\{1, \eta_2\}$	PROVED	Weingarten 16×16
Thm 5	$P = 2/3$ at aligned positions	PROVED	(b, c) recursion
Thm 6	$\eta_2 = 2/5$ is q-independent	PROVED	Weingarten q=2, 3, 4
Prop 7	α=β=η_2, p_{stat}=2/3	PROVED	Weingarten affine
Prop 6	$P(\ell$=$2) = 13/23$	PROVED	(b, c) recursion
—	$c_{\mathrm{VN}} = m^2 c_{\mathrm{RT}}$	MEASURED	6 depths, 10.34σ
—	$h_{S_2}^{\mathrm{qu}}$=14/45, $h_{S_2}^{\mathrm{ann}}$=8/27	MEASURED	N=300
—	$O(\varepsilon)$ straddle additive	PROVED	(b, c) at q=1+ε
Thm 8a	Standard Sector Projection	PROVED	Perm rep + tracelessness
—	2D closure: only $\{1, \eta_2\}$	PROVED	Thm 8a + hook characters
Thm 8	$h_{\mathrm{VN}} = w_2 \log_2(1/\eta_2)$	PROVED	Sections 6–7 reduction
Conj 2	$w_2 = 3/10$	OPEN	See Section 8

Table 6. Status of all results. Theorem 8 reduces the entropy-rate problem to Conjecture 2 (Lemma D); all other components are established exactly.

8 The Entropy-Rate Reduction and the Sector Weight Problem

8.1 Reduction to a single scalar quantity

The structural results of Sections 6–7 imply that the entropy-rate problem admits an exact and finite-dimensional reduction.

First, the Haar-averaged dynamics close in the two-dimensional operator space spanned by $\{\mathrm{SWAP}, \mathrm{TrTr}\}$. Second, the Standard Sector Projection Theorem (Section 8.2) establishes that, at all replica orders $q \geq 2$, only the trivial and standard representations of S_q contribute to the entropy-relevant observables. Third, in the limit $q \to 1+\varepsilon$, the straddle operation becomes additive rather than bilinear.

Together, these facts imply that the entropy rate is governed entirely by a single scalar parameter:

$$h_{\mathrm{VN}} = w_2 \cdot \log_2\left(\frac{1}{\eta_2}\right),$$

where w_2 is the weight with which the standard sector contributes to the von Neumann entropy in the replica limit.

Theorem 8 (Entropy-rate reduction). *Let $\eta_2 = 2/5$ be the Haar-averaged purity eigenvalue (Theorem 1). Then the quenched von Neumann entropy rate of the QFT boundary satisfies $h_{\mathrm{VN}} = w_2 \cdot \log_2(1/\eta_2)$, where w_2 is a scalar coefficient determined entirely by the projection of the standard sector of the transfer operator in the limit $q \to 1$. Moreover, the dynamics are fully confined to the two-dimensional sector $\{1, \eta_2\}$ (Theorem 8a), the recursion is exactly solvable in this sector, and w_2 is independent of tree depth, interval size, and recursion structure in the deep-tree limit. Thus the full entropy-rate problem reduces exactly to the evaluation of w_2.*

8.2 Standard Sector Projection Theorem

Theorem 8a (Standard Sector Projection). *At any integer replica order $q \geq 2$, the straddle perturbation $\delta\rho^{\otimes q}$ lies entirely in the standard representation of S_q. Consequently, only the eigenvalues $\{1, \eta_2\}$ of the transfer operator T_q contribute to the entropy rate h_q.*

Proof. The proof has three steps. (1) A single carry perturbation $\delta\rho$ enters one replica at a time through the tensor product $\rho^{\otimes(q-1)} \otimes (\rho + \delta\rho)$. This insertion generates the permutation representation of S_q, which decomposes as trivial $\oplus$ standard. (2) The perturbation is traceless: $\mathrm{Tr}(\delta\rho) = 0$. This kills the trivial component, leaving the perturbation entirely in the standard representation. (3) The Haar average and partial trace are S_q-equivariant maps, so they cannot regenerate the trivial component from a purely standard-sector input. As a corollary, the hook character theorem ($\chi_\lambda(q\text{-cycle}) = 0$ for non-hook representations λ) implies that only hook representations contribute to the q-cycle readout. Combined with step (2), this restricts the contributing sectors to $\{\text{trivial}, \text{standard}\} = \{\text{eigenvalue } 1, \text{eigenvalue } \eta_2\}$. □

This theorem has two immediate consequences. First, it provides an algebraic explanation for the numerically observed 2D closure: the straddle perturbation at $q = 3$ lies 100% in the η_2 eigenspace, and at $q = 4$ lies 99% in the η_2 eigenspace with the 1% residual in the 3/14 (non-hook) sector killed by the cycle readout. Second, it establishes that the entropy rate h_q at any replica order depends only on the two-dimensional $\{1, \eta_2\}$ sector, making the sector weight w_2 the sole unknown in the $q \to 1$ continuation.

We emphasize that this theorem controls only the sector support of the perturbation, not the entropy coefficient itself: the Standard Sector Projection proves that w_2 depends only on η_2, but does not evaluate w_2.

8.3 Structural constraints on w_2

The preceding sections impose strong constraints on the value of w_2:

1. **Spectral constraint.** Only the eigenvalues $\{1, \eta_2\}$ contribute to the entropy rate.
2. **Normalization constraint.** The trivial sector does not contribute due to tracelessness of the perturbation.

3. **Additivity constraint.** The $q \to 1$ limit is linear in the perturbation, eliminating higher-order coupling terms.

4. **Stationarity constraint.** The boundary state entering the evaluation is drawn from the stationary distribution of the MPDO.

Together, these imply that w_2 depends only on η_2 and not on any additional microscopic details of the tree.

The sector weight w_2 admits two equivalent descriptions:
(a) *Representation-theoretic form.* $w_2 = d/dq\big|_{q\to 1} \operatorname{Tr}(W_q \cdot M)$, where W_q is the q-cycle permutation acting in the permutation representation of S_q, and $M = b \cdot \mathrm{SWAP} + c \cdot \mathrm{TrTr}$.
(b) *Replica limit.* $d/dq\big|_{q=1}$(standard-sector straddle readout at replica q) $= 1 - \eta_2$.

A natural third candidate—the modular pairing $\alpha = -(1/\ln 2)\operatorname{Tr}(X_{\mathrm{std}} \cdot \log \rho_*)$ with ρ_* the stationary boundary state—is not well-posed as stated, because the Haar-averaged stationary state is $\rho_* = I/2$ (maximally mixed), making the pairing vanish identically for any traceless X_{std}. The entropy rate is an intrinsically second-order (quenched fluctuation) effect, not a first-order perturbation around the mean state. This is the fundamental reason the identity resists standard linearization techniques. We note that the entropy rate arises from the distribution of per-realization boundary states, not from any property of the Haar-averaged mean state.

8.4 Conjectured value

These constraints, together with numerical evidence, lead to the conjecture:

Conjecture 2 (Lemma D). $w_2 = (1 - \eta_2)/2 = 3/10$.

Substituting into the entropy-rate formula yields

$$h_{\mathrm{VN}} = \frac{1-\eta_2}{2} \cdot \log_2\left(\frac{1}{\eta_2}\right),$$

equivalently $c_{\mathrm{VN}} = m^2 \cdot c_{\mathrm{RT}}$ with $m^2 = 3/10$.

8.5 Numerical validation

The conjectured value is supported by high-statistics measurements across multiple independent tree depths:

$$c_{\mathrm{VN}} = 1.18845 \pm 0.00116 \text{ bits},$$

corresponding to $\beta_1 = c_{\mathrm{VN}}/(3c_{\mathrm{RT}}) = 0.09989$, consistent with $1/10$ within 1.1σ and separated from the nearest rational alternative by 10.34σ. This agreement holds uniformly across all tested depths and interval scales.

8.6 Nature of the remaining problem

Despite its finite-dimensional formulation, the evaluation of w_2 presents a genuine analytical difficulty.

The key feature is that the entropy rate is a **quenched average of a nonlinear functional**:

$$h_{\mathrm{VN}} = \mathbb{E}_V\left[S_{\mathrm{VN}}(\rho_\partial(V))\right],$$

where $\rho_\partial(V)$ is a random boundary MPDO. The logarithm couples all moments of the eigenvalue distribution, and the relevant distribution is neither asymptotically free nor accessible through standard large-N techniques.

The following approaches were investigated:

- **Weingarten moment expansion.** Provides exact results for integer Rényi entropies ($q \geq 2$) but does not extend to the $q \to 1$ limit, because the analytic continuation through the transcendental $\log_2(1/\eta_2)$ is not captured by rational replica data alone.
- **Free probability methods.** Apply to asymptotically free ensembles, whereas the MPDO tensors at bond dimension 2 are far from any large-N limit.
- **Replica symmetry breaking.** Designed for quenched disorder in classical spin systems, whereas the disorder here is in the unitary gates and the relevant order parameter is a quantum entropy.
- **Padé continuation from h_2.** The Rényi-2 rate $h_2 = 14/45$ is proved exactly, but the Padé continuation is underdetermined by one constraint because for qubits, h_3 is algebraically slaved to h_2 (the identity $\mathrm{Tr}(\rho^3) = (3\,\mathrm{Tr}(\rho^2) - 1)/2$ for qubits eliminates $q = 3$ as an independent data point).
- **Local channel expansions / Kubo–Mori quadratic form.** Capture single-step behavior but fail to reproduce the global entropy rate accumulated along the tree.
- **Resolvent / renewal formulations.** Organize the recursion but reduce to the same scalar identity.

In each case, the obstruction is the same: the nonlinearity of the von Neumann entropy applied to a quenched random MPDO.

8.7 Interpretation

The reduction achieved in this work shows that the entropy-rate problem is not fundamentally a problem of recursion, geometry, or large system size. Instead, it is a problem of **entropy evaluation for a structured random matrix ensemble**.

The boundary MPDO is characterized by fixed bond dimension, known transfer spectrum, exact recursion algebra, and constrained representation-theoretic support. Despite this, no existing method yields a closed-form expression for its quenched von Neumann entropy rate.

This identifies a well-defined open problem at the interface of quantum information theory and random matrix theory. Its resolution would constitute a new result: a rare example of a closed-form expression for the quenched von Neumann entropy rate of a random MPDO.

8.8 Status

All structural components of the entropy-rate problem have been established exactly: transfer spectrum, recursion closure, sector restriction, MPDO representation, and entropy-rate reduction.

The remaining step is the evaluation of a single scalar quantity w_2.

The conjectured value $w_2 = (1 - \eta_2)/2 = 3/10$ is strongly supported numerically (10.34σ) but remains analytically open.

8.9 Summary

The entropy-rate problem in the Quantum Family Tree model reduces exactly to the evaluation of a single scalar quantity w_2. All dependence on system size, recursion depth, and geometry has been eliminated.

The conjecture $w_2 = (1 - \eta_2)/2$ is consistent with all structural constraints and numerical evidence. Its proof would complete the entropy-rate theorem and provide a closed-form expression for the quenched von Neumann entropy rate of a random MPDO with known transfer spectrum.

9 Discussion

9.1 Geometry from genealogy

The Quantum Family Tree provides a concrete realization of a simple but powerful idea: spatial structure can emerge from genealogical relationships in a quantum process. In this model, distance is not imposed geometrically but is encoded in the depth of shared ancestry. Entanglement between boundary degrees of freedom reflects the amount of shared lineage, and the entropy of an interval measures the information separating two genealogical subtrees.

What distinguishes this construction from prior tensor-network models is that the geometry is not specified in advance and then populated with tensors. Instead, the geometry is *inherited* from the causal structure of a random unitary process. The Bruhat–Tits tree appears not as an input but as the natural organizing structure of the recursion.

9.2 Entropy as a global property of recursion

A central lesson of this work is that the entropy rate is not determined by local channel properties. Single-gate spectra do not contain the observed coefficient; single-qubit stationary states saturate and do not exhibit growth; local purity contractions overestimate the entropy rate. Instead, the entropy coefficient emerges only after global composition along the tree, and depends on the interplay between position averaging, carry propagation, and boundary crossings.

The two-chain MPDO provides the minimal object capturing this global structure. Its bond dimension remains fixed, yet it generates linear entropy growth through repeated injection of correlations at boundary crossings.

9.3 Exact solvability at the structural level

Despite the apparent complexity of the system, the underlying dynamics are highly constrained. This work establishes: closed-form closure of the recursion in a two-dimensional operator space; explicit determination of the transfer spectrum $\{1, \eta_2\}$; complete restriction of replica contributions to a two-dimensional sector; and identification of the boundary state as a two-chain MPDO.

These results eliminate all dependence on large system size or asymptotic limits. The entropy-rate problem is not an infinite-dimensional problem—it is a finite-dimensional one.

This type of reduction is atypical for random quantum systems, where complexity generally grows with system size. Here, the opposite occurs: complexity collapses to a minimal representation.

9.4 The sector weight as the central object

The reduction achieved in this work identifies a single quantity, w_2, as the carrier of all remaining complexity. This quantity determines the entropy rate, depends only on the standard-sector projection, and is independent of all geometric and combinatorial details.

The problem of computing the entropy rate is therefore equivalent to evaluating a specific readout of the stationary boundary state in the von Neumann limit. This isolates a new and well-defined object: the quenched von Neumann entropy rate of a constrained random MPDO. Unlike traditional random matrix ensembles, this object is neither large-dimensional nor asymptotically free, not governed by classical disorder averaging, and not reducible to moment data alone. It occupies an intermediate regime between exactly solvable models and fully chaotic ensembles.

9.5 Relation to holography

The relation $c_{\mathrm{VN}} = m^2 \cdot c_{\mathrm{RT}}$ mirrors the structure of holographic entropy formulas, where quantum corrections modify classical geometric contributions. In the present model, c_{RT} arises from the combinatorial geometry of the tree, and m^2 arises from the spectral gap of the transfer operator.

The equality of the GHPP expression and the transfer-gap expression at $\eta_2 = 2/5$ provides a nontrivial consistency between holographic and information-theoretic interpretations. Unlike continuum AdS/CFT, however, the entire structure here is explicit and finite. This makes it possible to isolate precisely which components are geometric, which are dynamical, and which remain unresolved.

9.6 Why the remaining step is nontrivial

At first glance, the remaining problem appears elementary: evaluate a readout involving a logarithm in a two-dimensional sector. However, this appearance is misleading. The stationary state is not a fixed matrix but the output of a random MPDO, and the entropy involves a quenched average over this ensemble.

The difficulty arises from three interacting features: the nonlinearity of the logarithm, quenched averaging over random unitary realizations, and finite bond dimension which precludes asymptotic simplifications. Standard tools each fail for specific reasons: moment methods do not capture the logarithm; replica methods lose constraints near $q = 1$; free probability does not apply at fixed dimension. The obstruction is therefore structural rather than technical.

9.7 Open directions

The reduction achieved here suggests several concrete directions:

1. *Sector weight evaluation.* Prove $w_2 = (1 - \eta_2)/2$ using representation-theoretic or operator-algebraic methods.

2. *General bond dimension.* Extend the analysis to $U(D)$ gates. The transfer gap identity $m^2(D) = (D{-}1)/(2(D{+}1))$ with $\eta_2(D) = 2/(D{+}1)$ extends to arbitrary $U(D)$ gates. The (b, c) recursion generalizes by replacing the SWAP and TrTr operators with their D-dimensional analogues SWAP_D and TrTr_D. The recursion remains 2-component because the Haar second moment for $U(D^2)$ still produces exactly two nonzero eigenvalues in the carry transfer matrix: $\{1, \eta_2(D)\}$.

3. *Geometric precondition.* The first law $\delta\langle H_A\rangle/\delta S = 1$ is proved exactly at $k{=}1$, where the modular Hamiltonian has rank 2 (aligned) or rank 4 (straddling). Extension to $k > 1$ requires the full MPDO modular Hamiltonian structure. For the Lewkowycz–Maldacena gravitational derivation to apply at all k, a geometric precondition ($D \gg \ell$) must be verified.

4. *Exact Rényi-2 slope.* Derive a closed-form expression for the annealed Rényi-2 rate (converged value 0.2961 per doubling).

5. *Replica structure.* The q=3 transfer matrix contains $m^2 = 3/10$ as a direct spectral value. Understanding why the q=3 spectrum encodes the $q \to 1$ entropy rate may illuminate the sector weight computation.

6. *Broader ensembles.* Identify other random tensor constructions admitting finite-dimensional entropy reductions.

10 Conclusion

We have introduced the Quantum Family Tree, a model in which spatial structure emerges from genealogical relationships in a random quantum process. In this framework, entanglement encodes kinship, and entropy measures separation between lineages.

The primary result of this work is a complete structural reduction of the entropy-rate problem. We show that the dynamics reduce exactly to a two-dimensional operator space; the transfer spectrum is determined in closed form; the boundary state is a two-chain MPDO with fixed bond dimension; and the entropy rate depends on a single scalar quantity w_2.

This reduction eliminates all dependence on system size and recursion depth, transforming a priori complex dynamics into a finite-dimensional problem.

The remaining step is the evaluation of w_2, which we formulate as an explicit conjecture (Lemma D). The conjectured value $w_2 = (1 - \eta_2)/2 = 3/10$ is strongly supported by numerical evidence (10.34σ) but remains analytically open.

The work therefore achieves a precise separation: all structural aspects of the entropy-rate problem are solved, and the remaining difficulty is localized to a single, well-defined quantity.

This identifies a new problem in quantum information theory: the computation of the quenched von Neumann entropy rate of a structured random MPDO. Its resolution would complete the present program and provide a rare example of an exact entropy formula in a nontrivial random quantum system.

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Code availability

gram_recursion_v6.py, bc_recursion.py, position_recursion.py, renyi2_highstat.py, mpdo_transfer_exact.py, two_chain_v2.py, and mpdo_vn_full.py are available on request.

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